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ABSTRACT

A model for fractions derived by an operation of "scattering into subunits" is hypothesized. Five stages in the model are discussed. Assumptions underlying the development of the model, features of the model, and disadvantages of the model are listed. (DT)

A Proposed Model for Teaching and Learning Common Fractions, and the Operations of Multiplication and Division of Fractions

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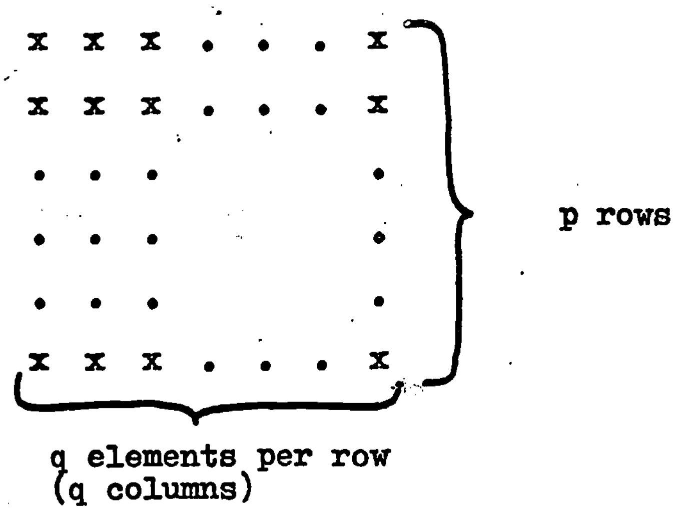
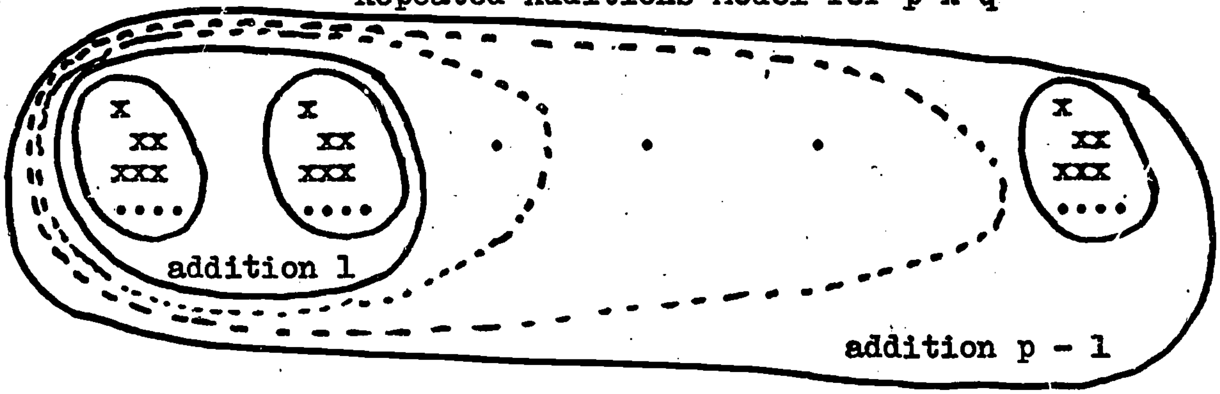
In the model proposed below a fraction is considered to be an answer to the question "p = q x ?" where p and q are counting numbers with q nonzero. Defining a fraction in this manner we must assume that the child has an understanding of multiplication of counting numbers, and that, in some sense, we can describe that understanding. Thus the model for fractions presupposes a model for multiplication of counting numbers. The two most common models for this operation are the Cartesian Product Model and the Repeated Additions Model. These models are illustrated below.

Cartesian Product Model for p x q



Paper prepared for GCSLTM workshop on Models for Teaching and Learning Mathematics, Atlanta, Georgia, April 24, 1976

Repeated Additions Model for p x q



q elements per pile

p piles

An essential feature of each of these models for multiplication is the equivalence (as sets and as displays) between pairs of rows, pairs of columns, or pairs of piles. We shall refer to this equivalence as an isomorphism.

A second feature of each of these models is the existence of an (unchanging) unit, and identification of this unit with the number 1. In order to extend this model to fractions we must assume a willingness on the part of the student to accept a set or conglomerate as a unit. This assumption does not seem unreasonable; in both models for multiplication there seems to be a necessity to consider sets as "hyper-units." Both models seem to consist of the following sequence of operations:

	Unit	$\xrightarrow[\text{collect}]{}$	Hyper-unit	$\xrightarrow[\text{collect}]{}$	Product
Cartesian Product:	Unit	$\xrightarrow[\text{collect}]{}$	row	$\xrightarrow[\text{collect}]{}$	Array
Repeated Additions:	Unit	$\xrightarrow[\text{collect}]{}$	pile	$\xrightarrow[\text{collect}]{}$	Pile of piles

The Model

The model hypothesized below is derived by reversing the operation of collection of units into hyper-units; the reverse operation will be called "scattering into subunits." We demand that the role of isomorphism and the idea of an unchanging unit be retained.

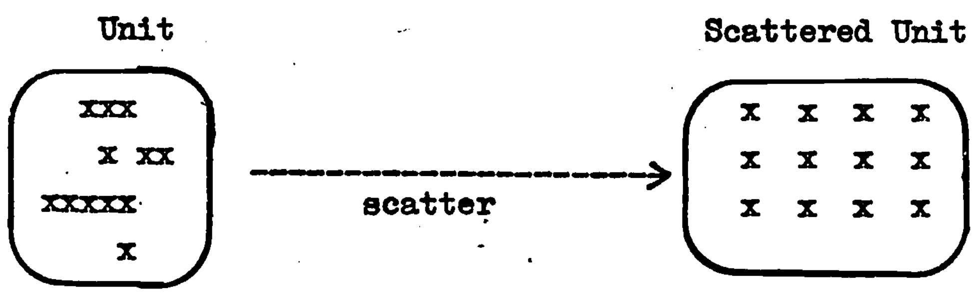
Beginning with the fraction 1/p, the model is developed through five stages at the "special units" level. A special unit for a fraction p/q is one which is readily viewed as containing kq isomorphic subunits (k a whole number). It is assumed that once each stage is internalized at the special units level, the student can move to a higher "general units" level; the mechanism for such movement could probably be based on deepening or abstracting the scattering operation.

The five stages in the model are associated with the following five questions:

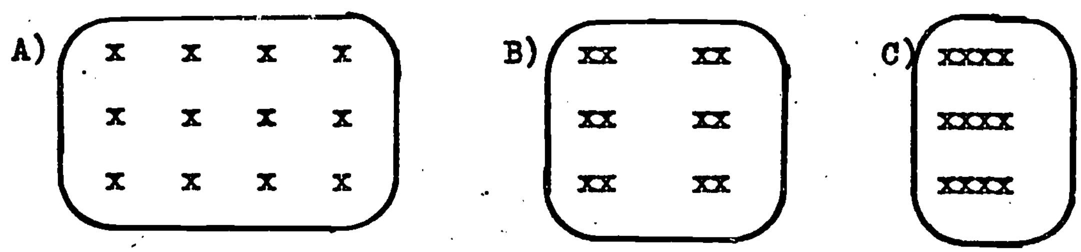
1) p x ? = 1
2) p x ? = q
3) p/q x r/s = ?
4) p/q x ? = 1
5) p/q x ? = r/s

The model can be based on either the Cartesian Product or Repeated Additions Model for counting number multiplication. The following development is in terms of the Cartesian Product model.

Stage 1. Given the problem "p x ? = 1" and a special unit, the first effort of the student is to try to bring the problem into conformity with his prior knowledge of multiplication of counting numbers. This can be accomplished best by arranging the unit to resemble the product model. We shall call this process "scattering;" note that the unit is retained during scattering.

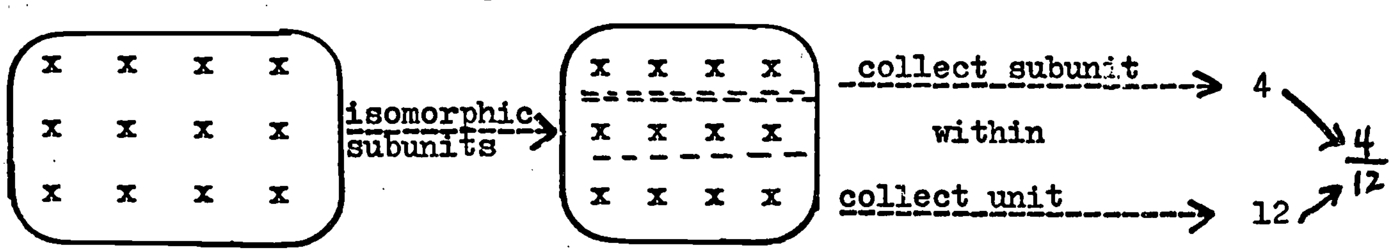


The particular scattering performed is determined in part by the number p, and in part by the student's sophistication or whim. Each of the following scatterings would be appropriate for the problem 3 x ? = 1 with a special unit of cardinality 12:

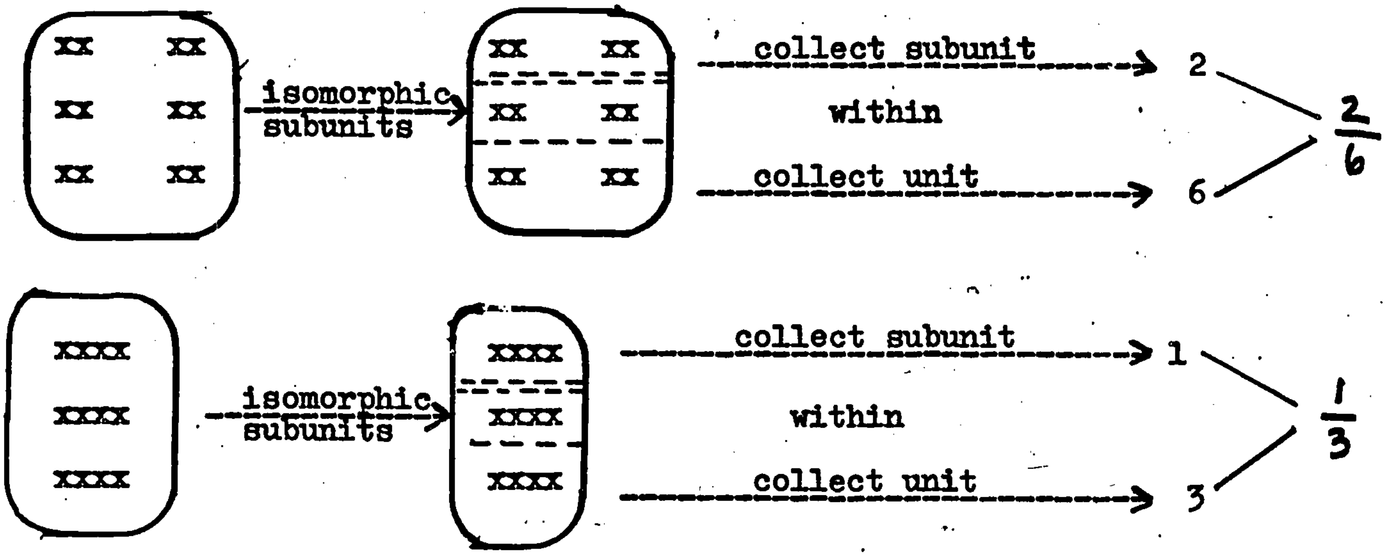


The scattering is done to facilitate formation of p isomorphic subunits. A pair of collection operations, together called "collecting subunit within unit" is then applied. The subunit collection determines the numerator, and the unit collection the denominator of the fraction.

Procedure for scattering A



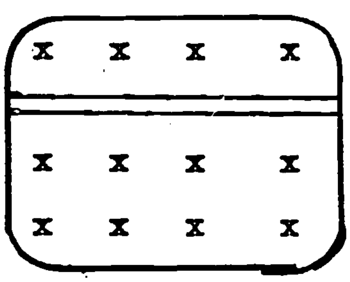
Similar procedures for scatterings B and C:

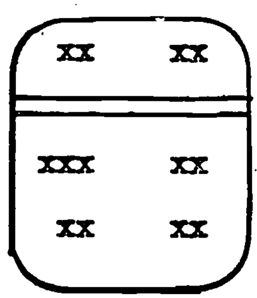


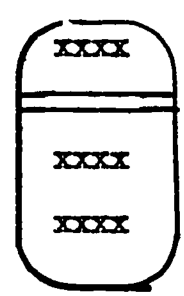
Thus the model for forming the answer to p x ? = 1 consists of the following steps:

1. Scatter unit
2. Determine p isomorphic subunits
3. Collect subunit within collected unit

Note that equivalent fractions arise from equivalent scatterings, with the reduced form of the fraction corresponding to the most efficient scattering. Natural representations of the fraction formed arise from steps 2 and 3; the representations are scattering-dependent and are illustrated below:

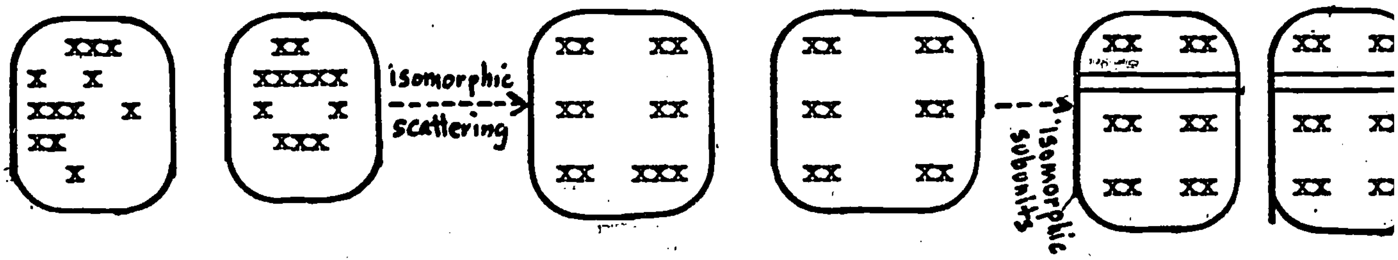
A) 

B) 

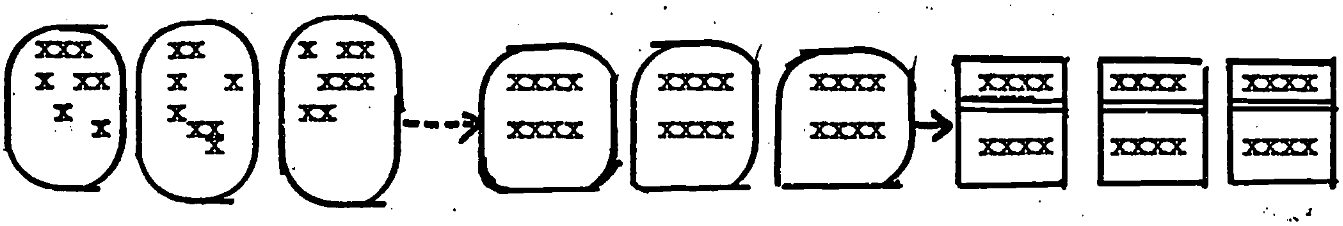
C) 

Stage 2. In order to solve the problem p x ? = q, the student begins with q isomorphic copies of the (special) unit, and proceeds as in stage 1. The q copies of the unit are scattered isomorphically, and isomorphic subunits are identified within each copy. The following diagram illustrates these procedures for the problems 3 x ? = 2 and 2 x ? =3.

3 x ? = 2; special unit of cardinality 12

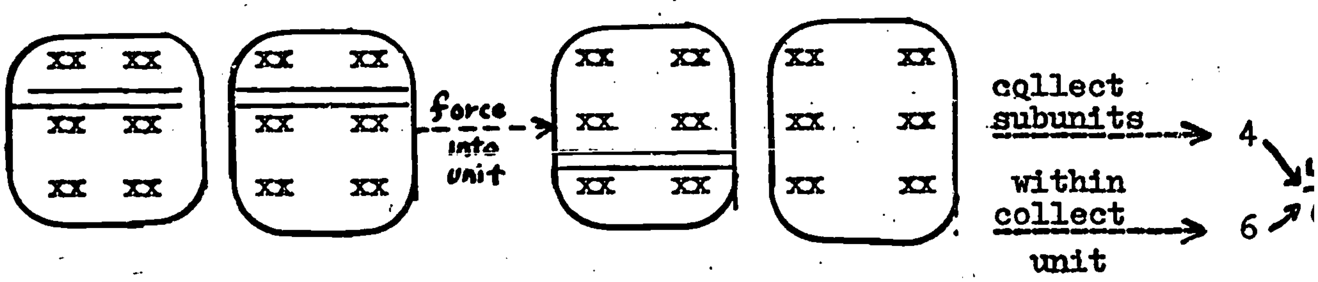


2 x ? = 3; special unit of cardinality 8

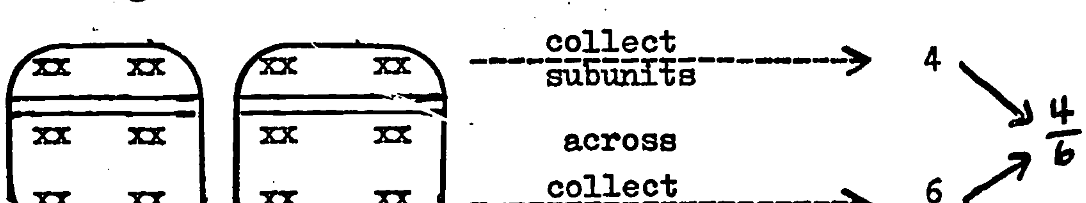


What remains is to collect within units. Two procedures are available: we can force subunits into units and then collect, or we can collect across within units.

Forcing for 3x ? = 2

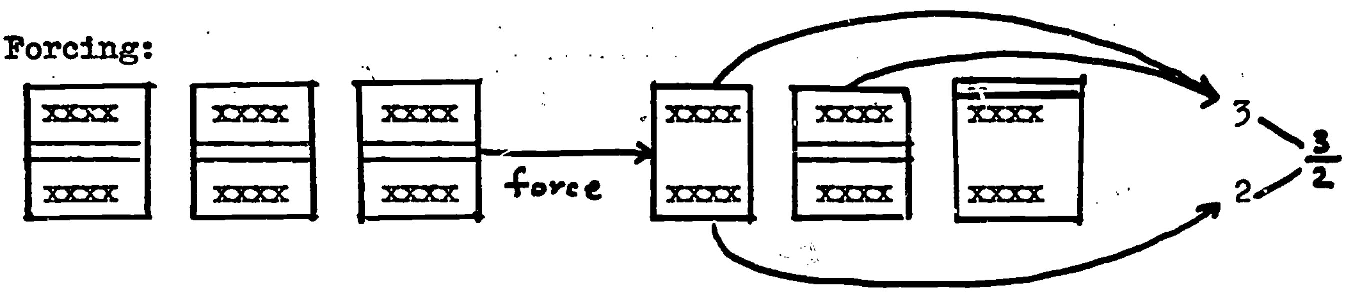


Collecting across for 3 x ? = 2

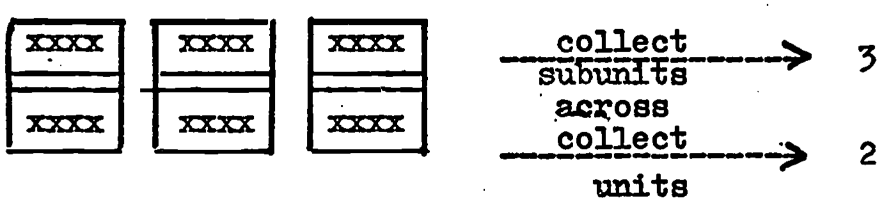


The forcing procedure has the advantage of giving a clearer representation of the fraction than the collecting across procedure when q is less than p. The situation becomes more complicated when p is less than q, especially as we wish to have a representation which can be manipulated in conjunction with operations. The following examples illustrate the two procedures for the problem $2 \times ? = 3$, with a special unit of cardinality 8.

Forcing:

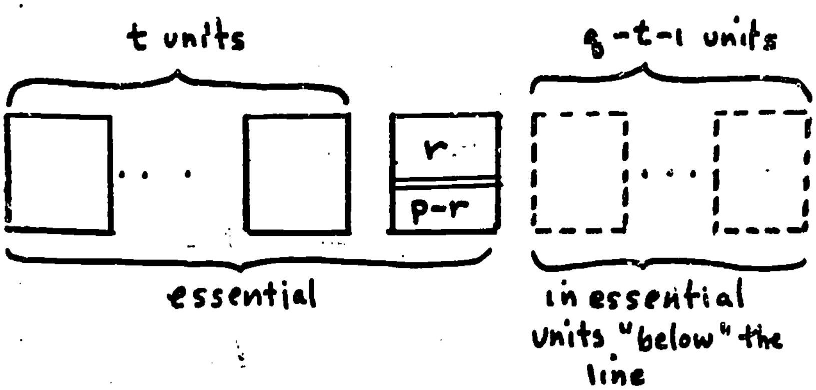


Collecting across:



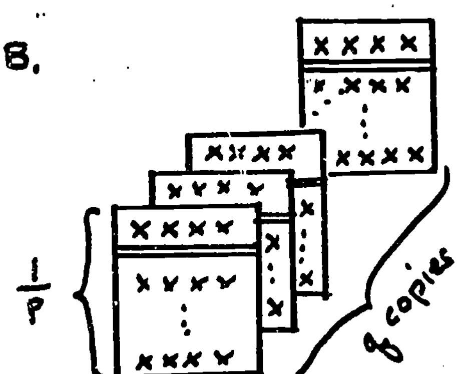
The representation of a fraction (greater than 1) which emerges from forcing would seem to be best made as in figure A below. The representation emerging from collecting across looks like a stack of representations for $1/p$, as in figure B.

A.



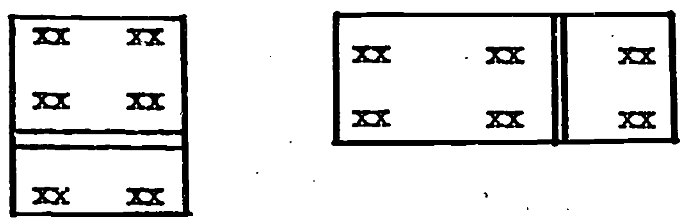
$q = pt + r,\ 0 \le r < p$

B.

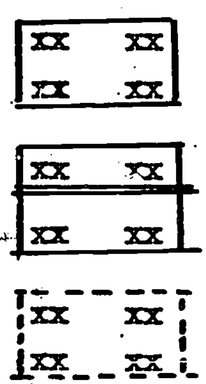


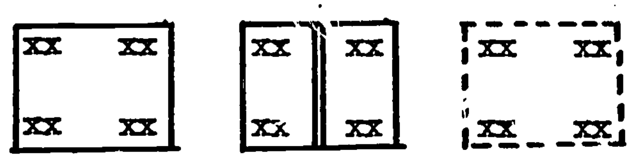
The stacking procedure can be used in the development of models for operations on fractions, and does have some intuitive appeal; however, it seems to require use of three dimensional displays or indexing systems. Therefore the rest of the model is developed using the forcing procedure.

In order to parallel the models for muptiplication of counting numbers as closely as possible, we extend the set of representations of fractions by adding transposes of the representations described above. Under this extension 2/3 could be represented by either of the displays:



Similarly, 3/2 could be represented by either of these:



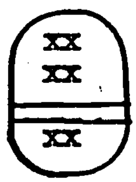


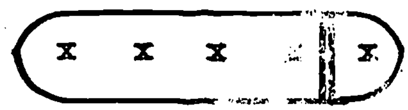
Note: Complete units below or to the right of the line can be deleted.

We are now ready to consider multiplication of fractions.

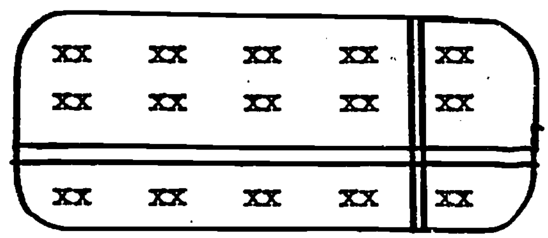
Stage 3. In developing the multiplication of fractions we mimic, as closely as possible, the Cartesian Product Model for the multiplication of counting numbers. In that model one factor is represented on a horizontal dimension and the other on the vertical dimension; the entire array is collected to form the product. In the multiplication of two fractions, the first fraction will determine the vertical representation and the second the horizontal representation; the combined representations determine a "crossed array." To determine the product of the fractions the upper left corner is collected within the (collected) unit. The following examples illustrate the procedure.

Example: 2/3 x 4/5

Model for 2/3: 

Model for 4/5: 

The crossed array: 2/3

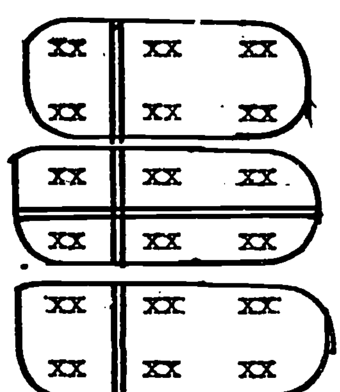


4/5

Collect upper left (8) within unit (15).

Example: 3/2 x 1/3

3/2

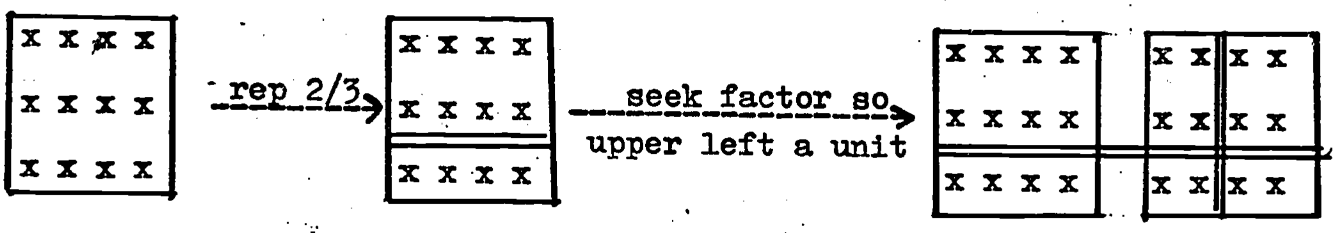


1/3

Collect upper left (3) within unit (6).

Stage 4. Division is the inverse of multiplication, and is developed in two stages: reciprocation and division of general fractions. For reciprocation we begin as in stage 1 with a special unit. This unit is used to determine the representation of p/q. The goal is then to use this unit as the vertical and find a crossed array with the unit in the upper left.

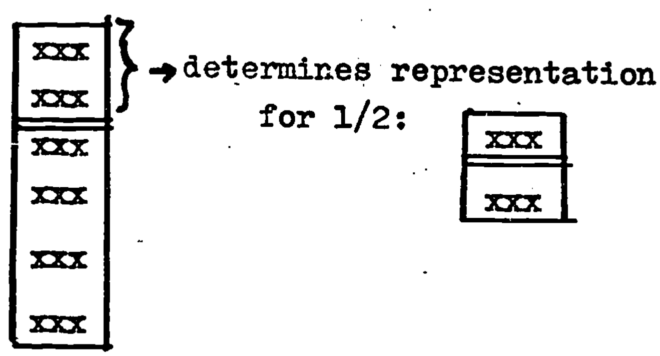
Example: 2/3 x ? = 1; special unit of cardinality 12



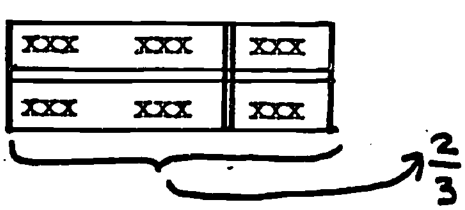
Stage 5. As in stages 1, 2, and 4, the process of division begins by determining a representative for the product. This representation is used to determine the representation of the known factor, and the other factor is then sought.

Example: 1/2 x ? = 2/6

Representation of 2/6:



Form crossed array to yield 2/6:



One of the main sources of students' errors in computation and reasoning with fractions seems to be related to confusion between multiplication and division of these numbers. The teaching model sketched above was developed in an effort to build an approach to teaching fractions which might avoid this confusion. In particular, it was desired that the representation of multiplication should not involve division of diagrams. The following considerations are related to this initial effort.

Assumptions underlying the development of the model

1. Fractions arise from consideration of division; therefore division and multiplication are more basic to fraction concepts, and should be taught first.

2. Insofar as fractions arise from consideration of part/whole relationships, the development of fractional concepts is dependent upon the ability to accept a set or conglomerate entity as a unit.

3. Either a Cartesian Product or Repeated Additions Model for counting number multiplication should underly a definition of fractions.

4. The key element of both the Cartesian Product and Repeated Additions Models for multiplication of counting numbers is the isomorphism of "units of units" (columns, rows, piles). This isomorphism should be retained, extended, interpreted, or redefined in the extension of the number system to include the fractions.

Features of the Model

1. Natural extension of the Cartesian Product or Repeated Additions Model for multiplication (and division) of counting

numbers; relies on extension of the underlying isomorphisms used in these models.

2. Multiplication and division are easily defined and distinguished within the model; the inverse relationship between these operations is reflected in the model.

3. Equivalent fractions occur naturally as alternate solutions to a problem.

4. Illustrates the meaninglessness of 0 as a denominator.

5. Utilizes part/whole relationships.

Disadvantages of the Model

1. Unwieldy and non-unique representations for fractions greater than 1.

2. Reliance on use of "special units."

3. Dependence upon visual representations.